

DOCUMENT RESUME

ED 367 538

SE 054 000

AUTHOR Fleener, M. Jayne; And Others
TITLE Integrating Mathematics with Ninth Grade Physical Science: The Proportionality Link.
PUB DATE Apr 93
NOTE 18p.; Paper presented at the Annual Meeting of the American Educational Research Association (Atlanta, GA, April 1993).
PUB TYPE Reports - Research/Technical (143) -- Speeches/Conference Papers (150)
EDRS PRICE MF01/PC01 Plus Postage.
DESCRIPTORS Abstract Reasoning; *Algorithms; *Cognitive Development; Formal Operations; Grade 9; High Schools; High School Students; *Interdisciplinary Approach; Learning Processes; Learning Strategies; Mathematics Instruction; Models; *Physical Sciences; Problem Solving; *Ratios (Mathematics); Science Activities; Science Experiments; Science Instruction
IDENTIFIERS Higher Order Learning; Lawson Test of Formal Reasoning; Learning Cycle Teaching Method; *Mathematical Thinking; Mathematics Education Research; *Proportional Reasoning

ABSTRACT

Higher order cognitive development and success in the study of high school mathematics and science require an understanding of rational number concepts and facility with proportional reasoning and computation. Proportional reasoning is an essential schema for developing formal operational thought. This study involving 16 ninth-grade students was conducted to investigate the following questions: (1) Does knowing a standard algorithm for solving proportion problems interfere with the development of proportional reasoning? (2) Are better mathematics students more flexible or intuitive than weaker students in applying proportional reasoning strategies to solve problems? and (3) What is the relationship between a student's general level of reasoning ability (concrete, transitional, or formal) and the strategies used for solving proportional reasoning tasks? Students engaged in proportional reasoning tasks and computational proportional problem solving in their science classes made a general positive gain in Lawson's Classroom Test of Scientific Reasoning scores, although gains for concrete learners were mixed. Explicit teaching of the concept of ratios and student engagement in exploratory studies of the relationships between and among ratios provoked development of proportional reasoning for average students. Students can benefit from experiences from which the cross-multiply-and-divide algorithm can be derived. (Contains 17 references.) (Author/MDH)

* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *

ED 367 538

Integrating Mathematics with Ninth Grade Physical Science:
The Proportionality Link

M. Jayne Fleener

University of Oklahoma

Susan L. Westbrook and Laura N. Rogers

North Carolina State University

"PERMISSION TO REPRODUCE THIS
MATERIAL HAS BEEN GRANTED BY

M. Jayne Fleener

TO THE EDUCATIONAL RESOURCES
INFORMATION CENTER (ERIC)."

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION
CENTER (ERIC)

X This document has been reproduced as
received from the person or organization
originating it.

Minor changes have been made to improve
reproduction quality.

Points of view or opinions stated in this docu-
ment do not necessarily represent official
OERI position or policy.

Running Head: Integrating Mathematics and Science

Paper presented at the Annual Meeting of the American Educational Research
Association (AERA), Atlanta, Georgia, April, 1993.

SE 004000

Abstract

Higher order cognitive development and success in the study of high school mathematics and science require an understanding of rational number concepts and facility with proportional reasoning and computation. Proportional reasoning is an essential scheme for developing formal operational thought. Few secondary students have had the experiences necessary for developing proportional thinking. Students lacking proportional, probabilistic, and correlational reasoning patterns do not choose to enroll in advanced mathematics and science courses requiring those abilities. A curriculum designed to enhance individual students' proportional reasoning would have far-reaching effects. By engaging students in laboratory activities in their science classes that reinforce mathematics concepts and develop proportional reasoning skills, mathematics and science learning and instruction may be strengthened. Before an integrated mathematics-science curriculum can be developed, however, more needs to be understood concerning students' facility with utilizing mathematics in their science classes. In science class, students may have difficulty applying their mathematical knowledge because of the different context or because underlying mathematical concepts have been learned in a rote or superficial way. Ultimately, the link the proportional reasoning construct provides may be more pivotal than as a tie between concrete-transitional and formal operational levels of thought. Proportional reasoning may be the key to the door that currently exists between high school and college mathematics and science. This study was conducted to investigate the following questions. 1. Does knowing a standard algorithm for solving proportion problems (like "multiply-then-divide") interfere with, or in some way inhibit, the development of proportional reasoning? 2. Are better mathematics students more flexible or intuitive when applying proportional reasoning strategies to solve problems? Conversely, do weaker mathematics students over-rely on set algorithms for solving proportion tasks? 3. What is the relationship between a student's general level of reasoning ability (concrete, transitional, or formal) and the strategies used for solving proportional reasoning tasks? Students engaged in proportional reasoning tasks and computational proportional problem solving in their science classes made a general positive gain in CTSR scores, although the gains for concrete learners were mixed. Explicit teaching of the concept of ratios and student engagement in exploratory studies of the relationships between and among ratios provoked development of proportional reasoning for average students. Science teachers can benefit from providing students with experiences from which the "cross-multiply-and-divide" algorithm can be derived.

Integrating Mathematics with Ninth Grade Physical Science: The Proportionality Link

Background

Higher order cognitive development and success in the study of high school mathematics and science require an understanding of rational number concepts and facility with proportional reasoning and computation (Behr, Harel, Post & Lesh, 1992; Heller, Ahlgren & Post, 1989; Lawson & Bealer, 1984; Lesh, Post & Behr, 1988; and Wollman & Lawson, 1978). Proportional reasoning is an essential scheme for developing formal operational thought (Inhelder & Piaget, 1958) which is highly correlated with achievement in science classes (Mitchell & Lawson, 1988).

Few secondary students have had the experiences necessary for developing proportional thinking (Lindquist, 1989; and NCTM, 1989). In ninth grade, where most students are taking a first course in algebra, only 10%-15% of the students can utilize the schema related to proportional reasoning (Lawson & Bealer, 1984; and Mattheis, et al., 1992). Moreover, students lacking proportional, probabilistic, and correlational reasoning patterns do not choose to enroll in advanced mathematics and science courses (i.e. Algebra II, Chemistry, and Physics) requiring those abilities (Lawson & Bealer, 1984).

A curriculum designed to enhance individual students' proportional reasoning would have far-reaching effects. By engaging students in laboratory activities in their science classes that reinforce mathematics concepts and develop proportional reasoning skills, mathematics and science learning and instruction may be strengthened. Before an integrated mathematics-science curriculum can be developed, however, more needs to be understood concerning students' facility with utilizing mathematics in their science classes.

A Learning Model for Proportional Reasoning

Research indicates that proportional reasoning develops through a series of increasingly complex levels (Harel, Guershon, Behr, Post, & Lesh, 1992; and Heller, Ahlgren, & Post, 1989). A model of epistemic proportional reasoning has been suggested (Fleener, 1993) based upon Vergnaud's (1988) four complexity variables (structure, context, numerical characteristics, and presentation) and Kieren's (1988) model of rational number knowledge building. Successful performance on varying proportional reasoning tasks depends upon situation-specific characteristics of the problem (such as presentation style or numerical difficulty) as well as contextual features (familiar or unfamiliar) and proportional characteristics (intensive versus extensive attributes) (Figure 1).

Insert Figure 1 about here

The proportional construct acts as a filter during the learning process as sense is made of experience (Figure 2). When problems are encountered for which the learner has no ready solution, the learner is placed in a state of conceptual conflict or disequilibrium. Accommodation of new strategies results in a change to the cognitive structure. Learners may, however, develop computational strategies which seem to resolve the conflict. Over-reliance on rote computational strategies may produce an over-generalization of solution strategies or what is known as "buggy errors". In science class, students may have difficulty applying their mathematical knowledge because of the different context or because underlying mathematical concepts have been learned at a superficial level. In addition, the focus of mathematics is different from science which may make communication of concepts and ideas difficult across content areas. Ultimately, the link the proportional reasoning construct provides may be more pivotal than as a tie between concrete-transitional and formal operational levels of thought. Proportional reasoning may be the key to the door that currently exists between high school and college mathematics and science. Unlocking the door is the necessary first step for developing an integrated mathematics and science curriculum.

Insert Figure 2 about here

Objectives of Study

Based upon the above understanding of the learning process and the pivotal role proportional reasoning plays in mathematics and science learning in high school, this study was conducted to investigate these three questions.

1. Does knowing a standard algorithm for solving proportion problems (like "multiply-then-divide") interfere with, or in some way inhibit, the development of proportional reasoning?

2. Are better mathematics students more flexible or intuitive when applying proportional reasoning strategies to solve problems? Conversely, do weaker mathematics students over-rely on set algorithms for solving proportion tasks?

3. What is the relationship between a student's general level of reasoning ability (concrete, transitional, or formal) and the strategies used for solving proportional reasoning tasks?

Study Procedures, Design and Analyses

The study was conducted in April, 1992 as part of a year-long project to investigate the relationships among science concept learning and mathematical process and content skills. Prior to the balance investigation, students had participated in learning cycles (Lawson, Abraham, & Renner, 1989) for developing concepts associated with minerals and rocks, erosion and weathering, electricity and circuits, density and flotation, conductivity of liquids, and basic chemical reactions (Atkinson, et al, 1988).

Subjects. Ten females and six males from two ninth grade physical science classes volunteered to participate in the study. Eight of the ten females were concurrently enrolled in Algebra I while the other two were taking geometry. Three of the males were in Algebra I, two in geometry, and one in pre-algebra. Students' ages ranged from 15 years, 7 months to 16 years, 3 months. The school involved in this study was a ninth and tenth grade mid-high with an enrollment of approximately 1000 students, located in a suburban midwestern university town.

Assessments and Interventions. The assessments and interventions occurred during a learning cycle (exploration, invention, expansion) investigation of the concept "balance on a lever." A qualitative, multiple-assessment design was utilized to determine (a) the relationship between a student's general level of reasoning ability (concrete, transitional, or formal) and the strategies used for solving proportional reasoning tasks; (b) whether knowing a standard algorithm for solving proportion problems (the "cross-multiply-then-divide" algorithm) interferes with or in some way inhibits the development of proportional reasoning; (c) whether better mathematics students are more flexible or intuitive than weaker students when applying proportional reasoning strategies to solve problems; and (d) whether weaker mathematics students over-rely on set algorithms for solving proportion tasks

Assessment instruments (developed by the research team) were administered prior to exploration and concept invention, and after the expansion phase of the balance investigation. A retention test was administered six weeks after the conclusion of the balance unit. Lawson's (1987) *Classroom Test of Scientific Reasoning* (CTSR) was given at the end of first semester in early January and at the end of the school year approximately six weeks after the balance investigation. The CTSR was used to determine student developmental levels and track student performance on the proportional reasoning items (3 & 4) found on that test. Semester CTSR scores indicated that 2 of the 16 students were

concrete, 10 were transitional, and 4 were formal operational before the balance investigation at the end of first semester.

The pre-balance-investigation measure (pretest) included proportional reasoning tasks ranging from discrete quantity ratio proportions, intensive quantity ratio proportions, and compensatory (inverse) proportions. Students were asked whether they had worked problems (a) this week, (b) this month, (c) this semester, (d) this year, or (e) not at all, of either of the forms:

$$a + b = c + d \quad \text{or} \quad \frac{a}{b} = \frac{c}{d}$$

to determine whether prior knowledge of a set algorithm may inhibit development of the proportional reasoning construct.

Students then manipulated a balance apparatus (a plank and a coffee can) in laboratory groups to generate data concerning the relationship between the weights placed on opposite sides of a fulcrum and the distance of the weights from the fulcrum. Aggregate data were collected from the laboratory groups and a teacher-led discussion examined the data from which students "invented" the mathematical equation for balance ($W1 \times D1 = W2 \times D2$). Two class periods were devoted to explorations of fractions, ratios and proportions as students used concrete materials such as candy pieces and paper clips to solve proportion problems. The expansion activity for the balance investigation then challenged the students to use the balance equation to determine the weight of unknown objects using the balance apparatus and a limited number of weights.

Ten days after the pretest was given students took a post-expansion (posttest) balance test. The posttest was structurally equivalent to the pretest. Numerical characteristics were varied to determine the effect of problem features, and contextual changes provided information concerning algorithmic reliance.

Six weeks after the balance investigation, students took a comprehensive final exam which included balance and proportional reasoning questions. The exam had two parts: a multiple choice section and a laboratory practical. Three questions on the multiple choice portion of the exam dealt with the balance concept. The practical exam had eleven laboratory stations set up throughout the room. Students cycled through the laboratory stations individually. Three stations were devoted to proportional reasoning and/or balance activities. Qualitative analyses of student written responses on the pretest, posttest, and final exams were examined to explore patterns in solution strategies and relationships among various contextual and individual features. This was not a controlled

study, so generalizations based on our analyses cannot be substantiated with this small sample of volunteers. Results may indicate, however, needed areas for further research and may suggest strategies for developing proportional reasoning and integrating mathematics and science at the high school level.

Results

Does knowing a standard algorithm for solving proportion problems (like "cross-multiply-then-divide") interfere with or in some way inhibit the development of proportional reasoning? It has been suggested that rote knowledge of the "cross-multiply-and-divide" algorithm, (hereafter referred to as the CMD algorithm), may interfere with the development of proportional thought (Lesh, Post, & Behr, 1988). Students were asked on the pretest whether they had seen proportion problems in either horizontal or vertical forms. Problems given were:

$$\frac{6}{8} = \frac{x}{24} \quad \text{and} \quad 6 \div 8 = x \div 24$$

Forced response options were (a) this week, (b) this month, (c) this semester, (d) this year, or (d) not at all.

Of the sixteen students participating in this study, twelve recalled working vertical oriented proportion problems while four did not. More specifically, one student had never solved proportion problems presented vertically, three didn't recall solving proportion problems represented horizontally, and one did not recall solving problems represented by either format. The remaining 11 students had all worked proportion problems presented in both formats.

Students then were asked to solve three computational proportion problems presented in a vertical format:

$$\text{a. } \frac{8}{12} = \frac{x}{18} \quad \text{b. } \frac{x}{12} = \frac{3}{2} \quad \text{c. } \frac{3}{x} = \frac{1}{2}$$

All but two of the twelve students who remembered solving vertically represented proportion problems used the CMD approach to solving the first problem. These ten students will be referred to as the CMD strategy group. Solution strategies for the next two problems varied because students could do those in their heads. Eight of the CMD approaches resulted in a correct answer. The two incorrect answers using the CMD algorithm were nonsensical. One of the students misapplied the algorithm by multiplying 12×8 while the other

started to set up the correct equation ($12x = ?$), then used a combination of strategies to find equivalent fractions. Her final answer was $12/18$ rather than 12. The two students in the group of twelve who did not solve the first problem using the CMD algorithm simply wrote correct answers without showing any work. It is assumed they used equivalent fractions ($8/12 = 2/3 = 12/18$) in their heads but without direct questioning it is hard to determine the particular approach(es) they may have taken.

Of the four students who had never seen proportion problems presented vertically, three used the equivalent fraction strategy. Two of the three successfully answered $x = 12$. The fourth student used a combination of CMD and equivalent fractions strategies indicating a move towards construction of the CMD algorithm.

Students were then asked to solve the following problem:

A meter stick is balanced by placing a fulcrum at the natural balance point of the stick. A 200 gram mass is placed 20 cm to the left of the fulcrum.

- a. Draw a picture of the way you think the system looks at this point.
- b. Where would you have to place a 100 gram mass to bring the system back into balance? Show your work. Include any equations you use to find the answer.

Of the 10 students in the CMD group, one did not answer the balance question and two drew incorrect sketches of the system. The two incorrect sketches showed the weight 20 cm to the left of the fulcrum, but the system was drawn balanced. Of the nine students in this group who responded to part b of this question, only three came up with the correct answer. Two of the three set up proportions but crossed them out when answers were unreasonable. Correct answers were recorded with no work to indicate solution strategies. The third student who came up with the correct solution set up an equivalence relation for the weights ($100 = (1/2) \times 200$) then multiplied 20×2 for a correct solution of 40 cm indicating an understanding of the inverse relationship. None of the six remaining students who answered this question in the CMD group were able to develop correct computational strategies. Two answered "To the right" without indicating how far to the right the weight would have to be placed. One estimated that the weight would have to be about 60 cm to the right but provided no work. Another suggested "at very almost end other side." The solution of five was given by two of these six students. One of the two set up a direct proportion equation and appeared to use a CMD approach. Her solution looked like:

$$20n/20 = 100/20 = n = 5.$$

The other student answered "probably around 5 cm to the right" but showed no work. It is interesting to note that not one of these students gave the typical (incorrect) response of 10 cm. perhaps suggesting the role of experience and the lack of reliance on rote computational procedures for solving balance proportional reasoning problems.

Of the six students not in the CMD strategy group, three successfully reasoned through the problem. One wrote the equation " $20 \times 2 = 40$ " while the other two showed no work. None of the students in this group attempted a CMD approach to this problem. Two of the remaining three students did not attempt solutions while the third estimated 50 cm.

It is hard to say whether the CMD algorithm interfered with performance on the balance problem. It is clear that the non-CMD group did not develop a CMD approach to the balance problem while several students in the CMD group tried a CMD approach. It may be that tasks for which students have little experience may have produced more reliance on rote strategies for the CMD group.

Are better mathematics students more flexible or intuitive when applying proportional reasoning strategies to solve problems? Conversely, do weaker mathematics students over-rely on set algorithms for solving proportion tasks?

In order to answer this question, students were divided into two groups based on mathematics background. Students who were above grade level, taking geometry in ninth grade, were considered better mathematics students and placed in the high-ability group. Four students were currently enrolled in geometry, eleven were in algebra, and one was in pre-algebra. Student strategies for solving the following question were compared on the pretest, posttest, and final exam practical. Posttest and final exam questions were structurally similar to the problem below although the numbers were more difficult (11, 7, 5):

A 9 foot pole at the school football field has a shadow 6 feet long at 2:00 in the afternoon. At the same time on the field, another pole has a shadow of 8 feet.

- Do you think the second pole is more or less than 9 feet tall? Explain your answer.
- Draw a picture of the second pole and its shadow.
- Exactly how tall is the second pole? Show all your work.

Three of the four geometry students correctly indicated that the second pole would be taller than the first, set up proportions, and used the CMD algorithm to correctly derive the answer of 12 feet. Two of the three students also provided correct solutions on the posttest and laboratory practical. The third student set up

the problem correctly on the practical but made a computational error. On the pretest, the fourth geometry student indicated that the shadow would be less although she did not explain why. She drew an isosceles right triangle then labeled the legs 8 ft. It is possible that something she was doing in geometry interfered with her solution of this problem. On the posttest and final exam practical, she correctly answered that the second pole would be less than the first, set up the equation as a proportion, and correctly performed the calculations.

Of the eleven algebra students, three could not answer the pretest balance question; (two did not provide an answer to the question; and one could not provide a numerical solution but indicated he thought the shadow would be less because "a smaller object has a bigger shadow."); four of the eleven used additive operations to derive a pole height of 11 feet; and the remaining four students worked the problem as a proportion and gave the correct answer. The four students who correctly worked the pole problem on the pretest continued to have success working the problems on the posttest and practical exams.

The four students who used additive strategies to derive the incorrect answer of 11 on the pretest employed a variety of strategies on the posttest and practical exam. Darla correctly indicated the second pole would be less than the first on the posttest but set-up an incorrect proportion

$$\frac{11}{5} = \frac{x}{7}$$

with missing equals sign and denominators switched. She did, however, correctly solve the problem on the laboratory practical suggesting other experiences in class may have helped her recognize how to set up and solve proportion problems of this type. Kim correctly set up the proportions on the posttest and final but made a calculation on the practical portion of the final exam. Although Chris provided no work on the posttest to indicate the strategy she used to derive an answer of 9 feet, her solution on the laboratory practical indicated she persisted in an additive approach

$$\begin{aligned} 11 - 7 &= 4 \\ N - 5 &= 4 \\ N &= 9. \end{aligned}$$

Denise likewise persisted in an additive approach.

Of the three algebra students who could not answer the pole problem on the pretest, two of them made significant progress by the time of the final exam. Haley was able to solve the posttest and final problems using proportional

reasoning while Jake used an additive strategy on the posttest and a proportional approach on the final. Jason, the pre-algebra student, went from an additive response on the pretest, to a proportional approach on the posttest, and back to an

additive strategy on the final exam. It is clear he had expanded his repertoire of solution strategies but had not developed proportionality.

The geometry students as a group were more successful and consistent in their approach to solving the balance problem. They did not need to develop alternative strategies for solving the balance problem since they already had intuitive, ready solutions for the problem. The less capable mathematics students used a variety of solutions, but the solution strategy which resulted in a correct answer relied on the CMD algorithm.

What is the relationship between a student's general level of reasoning ability (concrete, transitional, or formal) and the strategies used for solving proportional reasoning tasks? First semester CTSR scores indicated that two of the sixteen students were concrete, ten were transitional, and four were formal operational before the balance investigation. By the end of the second semester, three scored at the concrete level, six scored at the transitional level, and seven were at the formal operational level. Two students' CTSR scores decreased in the four month period, while nine individual's scores increased and five stayed the same. Of interest is how students did on the proportional reasoning items of the CTSR after proportional reasoning instruction and participation in an investigation on balance.

In January, only three of the sixteen students could do either of the proportional reasoning tasks on the CTSR. By the May administration of the test, five students could perform both tasks. Three of the four geometry students began the semester testing at the formal operational stage although one of those fell to the transitional level by the end of the year. We saw above that the geometry students tended to have computational facility with proportional reasoning strategies and were consistently and correctly using the CMD algorithm. The top four algebra students likewise scored high on the CTSR and developed proportional problem solving expertise quickly.

The middle group of students who were transitional made some gains as solutions to the pole problem indicated. These students had the most changes in scores in both directions indicating that they were in transition and had not solidified their own thinking and problem solving strategies.

As might be predicted, the students who scored at the formal level had more success with the balance concept and proportional problem solving. They did not show changes in problem solving strategies because they were already

successful with problem solving.

Low scoring students on the CTSR tended to try more of a variety of problem solving strategies but did not seem to have the ability to select appropriate strategies. Implementation of proportional problem solving strategies likewise was met with a varying degree of success.

Conclusions

It may be that once the CMD algorithm has been constructed by students, it becomes a powerful and preferred solution strategy for proportional problem solving. Students who have rotely learned the CMD algorithm, however, seem to utilize a variety of solution strategies, many of which reveal buggy computational approaches to rational number understanding and an immature proportional reasoning construct. Students engaged in proportional reasoning tasks and computational proportional problem solving in their science classes made a general positive gain in CTSR scores, although the gains for concrete learners were mixed.

Significance

Higher order reasoning and problem solving require the development of proportional reasoning. Seldom are proportion tasks understood beyond the instrumental application of the "cross-multiply and divide" algorithm. Past attempts to "teach" proportional reasoning have included grouping (ratio) problem types and teaching rules and methods for problem solutions rather than focusing on developing proportional reasoning abilities. Taught in isolation, instrumental understanding of proportion concepts has not been transferred across content areas. Over-reliance on the CMD algorithm seems to interfere with proportional problem solving for students who have not invented the approach. Explicit teaching of the concept of ratios and student engagement in exploratory studies of the relationships between and among ratios provoke development of proportional reasoning for average students. Science teachers can benefit from providing students with experiences from which the CMD algorithm can be derived.

References

- Atkinson, L., Cate, J., Marek, E.A., Renner, J.W., Askey, D., Bowers, D., Brady, J., Gross, D., Kammerlocker, J., Privett, J., Sells, J., Sutton, S., & Thompson, S., (1988). Physical science: Level 9. Norman, OK: Norman Public Schools.
- Behr, M.J., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D.A. Arouws (Ed.), Handbook of research in mathematics teaching and learning (pp. 296-333). Reston, VA: National Council of Teachers of Mathematics.
- Fleener, M.J. (1993). Proportional reasoning of preservice elementary education majors: An epistemic model of the proportional reasoning construct. Paper presented at the Annual Meeting of the American Educational Research Association (AERA), Atlanta, Georgia, April, 1993.
- Harel, Guershon, Behr, M., Post, T., & Lesh, R. (1992). The blocks task: Comparative analysis of the task with other proportion tasks and qualitative reasoning skills of seventh-grade children in solving the task. Cognition and instruction, 9(1), 45-96.
- Heller, P.M., Ahlgren, A. & Post, T. (1989). Proportional reasoning: The effect of two context variables, rate type, and problem setting. Journal of research in science teaching, 26(3), 205-220.
- Inhelder, B. & Piaget, J. (1958). The growth of logical thinking from childhood to adolescence. New York: Basic Books.
- Kieren, T.E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In Hiebert, J. & Behr, M. (Eds.), Number Concepts and Operations in the Middle Grades. Reston, VA: National Council of Teachers of Mathematics, 162-181.
- Lawson, A.E. (1987). The Classroom Test of Scientific Reasoning. Arizona State University, Tempe Arizona.
- Lawson, A.E., Abraham, M.R., & Renner, J.W. (1989). A Theory of Instruction: Using the Learning Cycle to Teach Science Concepts and Thinking Skills. [Monograph]. National Association for Research in Science Teaching.

- Lawson, A.E. & Bealer, J.M. (1984). The acquisition of basic quantitative reasoning skills during adolescence: Learning or development? Journal of research in science teaching, 21(4), 417-423.
- Lesh, R., Post, T. & Behr, M. (1988) Proportional reasoning, in Number concepts and operations in the middle grades. Reston, VA: National Council of Teachers of Mathematics, 93-118.
- Lindquist, M.M. (Ed.) (1989). Results from the fourth mathematics assessment of the National Assessment of Educational Progress. Reston, VA.: National Council of Teachers of Mathematics.
- Mattheis, F.E., Spooner, W.E., Coble, C.R., Takemura, S., Matsumoto, S, Matsumoto, K., & Yoshida, A. (1992). A study of the logical thinking skills and integrated process skills of junior high school students in North Carolina and Japan. Science education, 76, 211-222.
- Mitchell, A. & Lawson, A.E. (1988). Predicting genetics achievement in non-majors college biology. Journal of research in science teaching, 25, 23-37.
- National Council of Teachers of Mathematics. (1989). Curriculum and evaluation standards for school mathematics. Reston, VA.: Author.
- Vergnaud, G. (1988). Multiplicative structures. In Hiebert, J. & Behr, M. (Eds.), Number Concepts and Operations in the Middle Grades. Reston, VA: National Council of Teachers of Mathematics, 141-161.
- Wollman, W. & Lawson, A.E. (1978). The influence of instruction on proportional reasoning in seventh graders. Journal of research in science teaching, 15(3), 227-232.

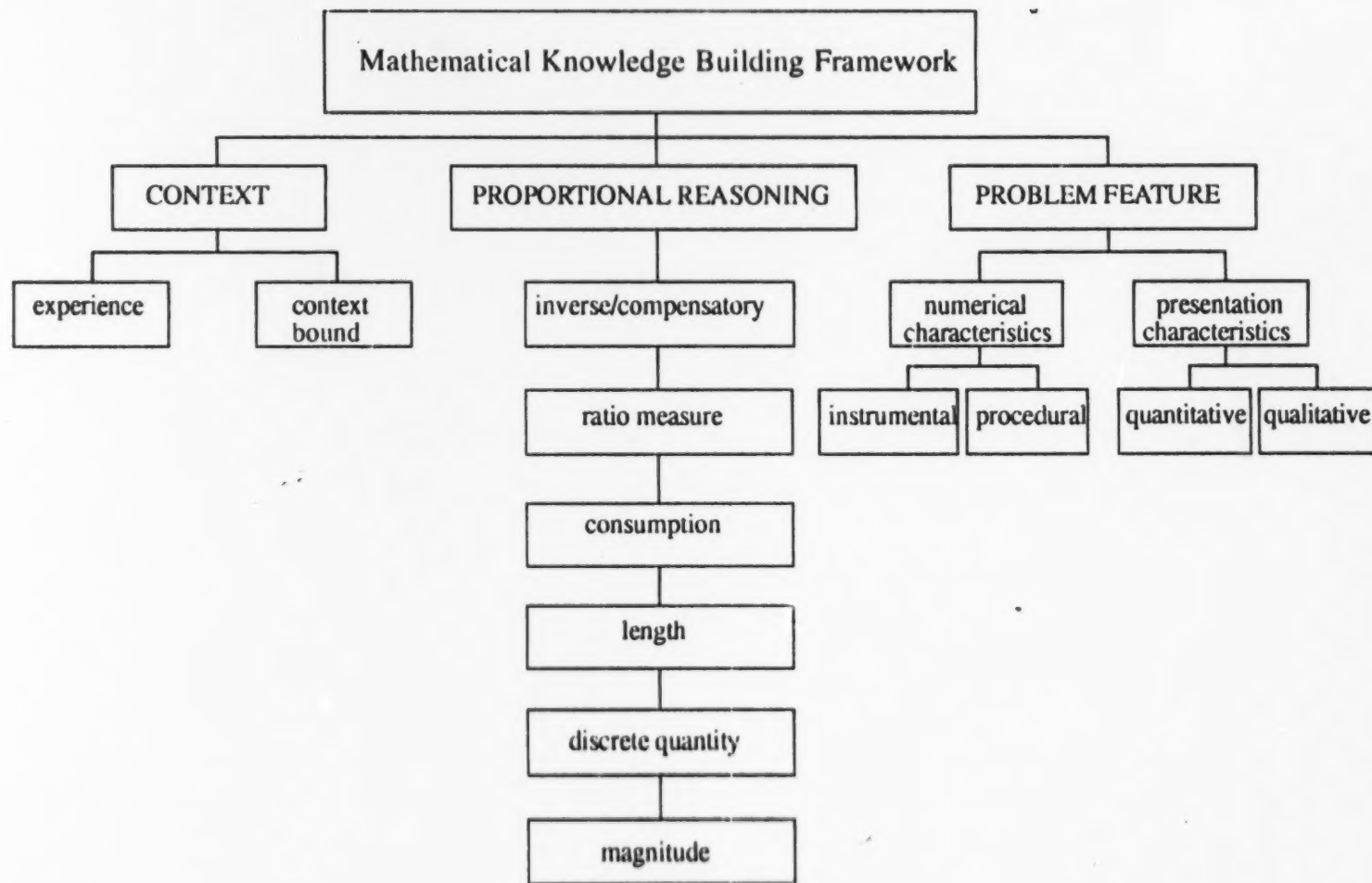


Figure 1. Proportional reasoning construct within a mathematical knowledge building framework

LEARNING: A GENERIC CONSTRUCTIVIST MODEL

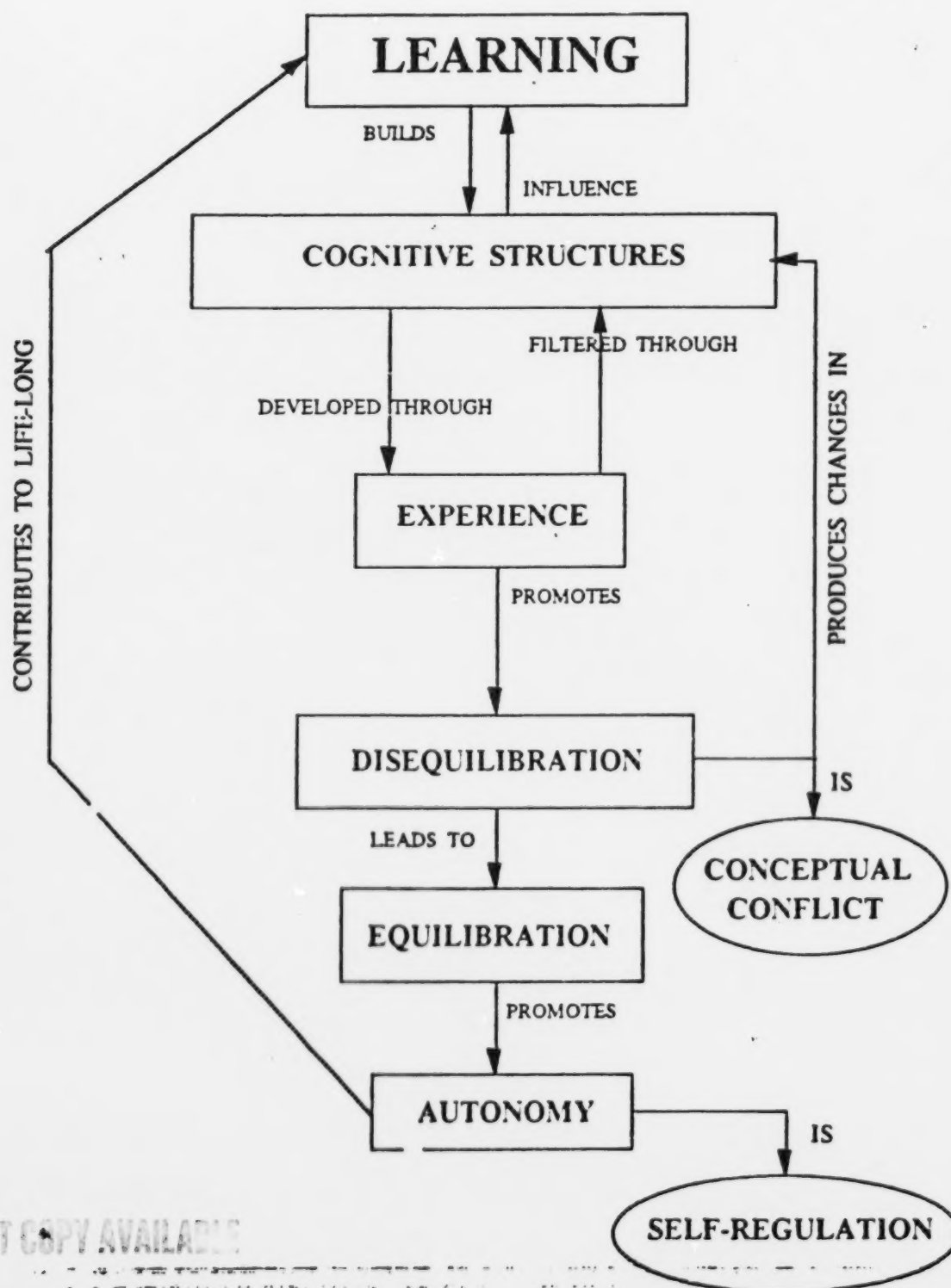


Figure 2. Learning: A generic constructivist model (reprinted by permission from Westbrook, Rogers, & Fleener, 1993). 18